

International Journal of Modern Physics E
 © World Scientific Publishing Company

GAUGE-INVARIANT SOFT MODES IN YANG-MILLS THEORY

HILMAR FORKEL

*Departamento de Física, ITA-CTA, 12.228-900 São José dos Campos, São Paulo, Brazil and
 Institut für Theoretische Physik, Universität Heidelberg, D-69120 Heidelberg, Germany*

Received (received date)

Revised (revised date)

A gauge-invariant saddle point expansion for the Yang-Mills vacuum transition amplitude on the basis of the squeezed approximation to the vacuum wave functional is outlined. This framework allows the identification of gauge-invariant infrared degrees of freedom which arise as dominant sets of gauge field orbits and provide the principal input for an essentially analytical treatment of soft amplitudes. The analysis of the soft modes sheds new light on how vacuum fields organize themselves into collective excitations and yields a gauge-invariant representation of instanton and meron effects as well as a new physical interpretation for Faddeev-Niemi knots.

The essence of the soft dynamics behind the most important QCD vacuum and hadron properties is expected to involve just a few types of gluonic long-wavelength modes. The quest for these infrared degrees of freedom (IRdofs) has focused primarily on semiclassical fields like instantons¹, monopoles^{2,3} and vortices⁴ and inspired the development of various vacuum models. All these field configurations are gauge-dependent, however, and thereby obstruct the analytical treatment and physical interpretation of their ensembles. Gauge-invariant or fully gauge-fixed formulations of the dynamics^{5,6}, on the other hand, generally involve nonlocalities and are therefore at least as difficult to handle analytically. Moreover, the underlying degrees of freedom typically receive contributions from a wide variety of gauge fields and therefore obscure the relation to gauge-dependent IRdofs.

Below we will outline an approach⁷ which circumvents the problems of dealing with individual gauge fields. Instead, it treats the contributions from their minimally gauge-invariant generalizations, i.e. their gauge orbits, jointly. This preserves traceable links to the gluon content and results in a representation by local matrix fields which gather universal contributions from dominant orbits to soft amplitudes. These collective gluonic IRdofs emerge as contributions to the vacuum overlap amplitude of $SU(N)$ Yang-Mills theory in the Schrödinger representation⁸,

$$Z' := \langle 0, t_+ | 0, t_- \rangle = \int D\vec{A} \Psi_0^* [\vec{A}, t_+] \Psi_0 [\vec{A}, t_-]. \quad (1)$$

2 Hilmar Forkel

Starting from an approximate wave functional

$$\psi_0^{(G)}[\vec{A}] = \exp \left[-\frac{1}{2} \int d^3x \int d^3y A_i^a(\vec{x}) G^{-1ab}(\vec{x} - \vec{y}) A_i^b(\vec{y}) \right] \quad (2)$$

of Gaussian type, we implement asymptotic freedom (in G^{-1}) and obtain the associated, gauge-invariant vacuum wave functional by integrating over the gauge group as

$$\Psi_0[\vec{A}] = \sum_n e^{iQ\theta} \int D\mu(U^{(Q)}) \psi_0[\vec{A}^{U^{(Q)}}] =: \int DU \psi_0[\vec{A}^U] \quad (3)$$

($d\mu$ is the Haar measure, Q the homotopy degree and θ the vacuum angle). Merits and potential shortcomings of the Gaussian ansatz (2) are discussed in Ref.⁷.

After interchanging the order of integration over gauge fields and group, factoring out a gauge group volume and evaluating the \vec{A} integral exactly, one ends up with a functional integral $Z = \int DU \exp(-\Gamma_b[U])$ over the “relative” gauge orientation $U \equiv U_-^{-1}U_+$ where the 3-dimensional Euclidean bare action⁹ is

$$\Gamma_b[U] = \frac{1}{2g_b^2} \int d^3x \int d^3y L_i^a(\vec{x}) D^{ab}(\vec{x} - \vec{y}) L_i^b(\vec{y}). \quad (4)$$

The U -dependence enters through the Maurer-Cartan forms $L_i = U^\dagger \partial_i U =: L_i^a \frac{\tau^a}{2i}$ and higher-order corrections to the bilocal operator $D^{ab} = \left[(G + G^U)^{-1} \right]^{ab} \simeq \frac{1}{2} G^{-1} \delta^{ab} + \dots$. Hence $\Gamma_b[U]$ gathers all contributions to Z whose approximate vacua ψ_0 at $t = \pm\infty$ differ by the relative gauge orientation U . The variable U thus represents the contributions of a specifically weighted gluon orbit ensemble to the vacuum overlap and is gauge-invariant by construction.

In order to access the physics which contributes to soft Yang-Mills amplitudes with external momenta $|\vec{p}_i|$ smaller than a typical hadronic scale μ , we now combine a renormalization group evolution of the bare action - to integrate out the UV modes with momenta $|k_i| < \mu$ explicitly - with a subsequent soft gradient expansion. The result is a local effective Lagrangian

$$\mathcal{L}(\vec{x}) = -\frac{\mu}{2g^2(\mu)} \text{tr} \left[L_{<,i}(\vec{x}) L_{<,i}(\vec{x}) + \frac{1}{2\mu^2} \partial_i L_{<,j}(\vec{x}) \partial_i L_{<,j}(\vec{x}) + \dots \right] \quad (5)$$

($g(\mu)$ is the one-loop coupling, $L_{<,i} = U_{<}^\dagger \partial_i U_{<}$ where $U_{<}$ contains only Fourier modes with $|k_i| < \mu$, and $\mu \simeq 1.3 - 1.5$ GeV) which has the form of a controlled expansion in powers of $(\|\partial_i U_{<}\|/\mu)^2$. For practical purposes we found the truncation at second order to yield a sufficient approximation (at the few percent level) to the full action. The locality and structural simplicity of the Lagrangian (5) is a benefit of reformulating the dynamics in terms of gauge-invariant soft-mode fields.

The above preparations enable us devise a practicable steepest-descent expansion for the functional integral $Z_{<}$ over the soft modes, based on the saddle points $\bar{U}_i(\vec{x})$ which solve

$$\left. \frac{\delta \Gamma[U_{<}]}{\delta U_{<}(\vec{x})} \right|_{U_{<} = \bar{U}_i^{(Q)}} = 0 \quad (6)$$

at fixed Q . The $\bar{U}_i^{(Q)}(\vec{x})$ represent the IRdofs we are looking for. Their contributions to soft amplitudes, including e.g. gluonic Green functions, are obtained by differentiating $Z_<$ with respect to suitable sources. The reliability of the leading-order approximation increases with their action value, parametrically enhanced by the factor $g^{-2}(\mu) \gg 1$ (for $\mu \gtrsim 1.3 - 1.5$ GeV), although systematic higher-order corrections can be calculated from fluctuations around them. The expressions for the action and saddle point equations in terms of (ϕ, \hat{n}) with $U_<(\vec{x}) = \exp[\phi(\vec{x}) \hat{n}^a(\vec{x}) \tau^a/2i]$ and $\hat{n}^a \hat{n}^a = 1$ for $N = 2$ are given in Ref.⁷.

Important generic properties of the IRdofs include their scale stability due to a virial theorem and three topological quantum numbers: a winding number $Q[U]$ (due to $\pi_3(S^3) = Z$), a monopole-type degree $q_m[\hat{n}]$ based on $\pi_2(S^2) = Z$ and finally a linking number $q_H[\hat{n}]$ in the Hopf bundle $\pi_3(S^2) = Z$ which classifies knot solutions. They entail the lower action bounds of Bogomol'nyi type

$$\Gamma[U] \geq \frac{12\pi^2}{g^2(\mu)} |Q[U]|, \quad \Gamma[\phi_k = (2k+1)\pi, \hat{n}] \geq \frac{2^{9/2} 3^{3/8} \pi^2}{g^2(\mu)} |q_H[\hat{n}]|^{3/4} \quad (7)$$

which ensure that contributions from saddle points in high charge sectors to soft amplitudes can generally be neglected.

While most saddle-point solutions have to be found numerically, several non-trivial analytical solution classes, e.g. of the type

$$\bar{\phi}^{(\hat{n}=c)}(r) = c_1 + \frac{c_2}{\sqrt{2}\mu r} \left(1 - e^{-\sqrt{2}\mu r}\right), \quad \hat{n}^a = \text{const.} \quad (8)$$

exist and further, particularly symmetric ones can be obtained by solving simplified field equations. Those include topological soliton solutions of hedgehog type $\hat{n}^a(\vec{x}) = \hat{x}^a$, $\phi(\vec{x}) = \phi^{(hh)}(r)$ whose Lagrangian reduces to

$$\mathcal{L}^{(hh)}(r) = \frac{\pi}{g^2(\mu)\mu} \left[\frac{1}{2} (r\phi'')^2 + (3 + \mu^2 r^2) (\phi')^2 + 4\mu^2 (1 - \cos \phi) \right]. \quad (9)$$

The hedgehog saddle points turn out to comprise mainly contributions from regions in A space around the classical solutions of the Yang-Mills equation, i.e. (multi-) instantons and merons, and were found numerically in Ref.⁷. (Hedgehog solutions with a monopole-type singularity at the origin also exist.) The one-instanton dominated solution, e.g., rather closely resembles its Yang-Mills counterpart

$$\phi_{I,YM}(r) = -\frac{2\pi r}{\sqrt{r^2 + \rho^2}} \quad (10)$$

with the same relative gauge orientation and $Q = 1$, but also contains quantum fluctuations. The latter stabilize the size of our solution at $\rho \simeq 2\mu^{-1}$ for $\mu \simeq 1.5$ GeV, compatible with instanton liquid model¹⁰ and lattice¹¹ results, and resolve the IR instabilities of classical Yang-Mills instanton gases. Our meron-type solutions share half-integer Q values and infinite action (due to angle-dependent asymptotics) with the pointlike Yang-Mills merons but remain nonsingular and acquire a finite size due to quantum fluctuations. Indications for a relatively large meron entropy,

4 Hilmar Forkel

potentially able to overcome their infinite-action suppression, are encountered as well. Moreover, we found solutions which have no obvious counterparts in classical Yang-Mills theory. One of the most intriguing classes consists of solitonic links and knots. Those emerge from a generalization of Faddeev-Niemi theory¹²,

$$\mathcal{L}^{(\phi_k)}(\vec{x}) = \frac{\mu}{g^2(\mu)} \left[(\partial_i \hat{n}^a)^2 + \frac{1}{\mu^2} (\varepsilon^{abc} \partial_i \hat{n}^b \partial_j \hat{n}^c)^2 + \frac{1}{2\mu^2} (\varepsilon^{abc} \hat{n}^b \partial^2 \hat{n}^c)^2 \right], \quad (11)$$

which turns out to be embedded in our soft-mode Lagrangian for fields of the form $\phi_k = (2k+1)\pi$ with $U_k(\vec{x}) = (-1)^k i\tau^a \hat{n}^a(\vec{x})$. Hence our approach provides a new dynamical framework and physical interpretation for Faddeev-Niemi-type knot solutions as gauge-invariant IR degrees of freedom whose underlying gluon field ensembles carry a collective Hopf charge.

In summary, we have developed a calculational framework for the Yang-Mills vacuum transition amplitude in the Schrödinger representation which reveals new, gauge-invariant infrared degrees of freedom. Some of them are related to tunneling solutions of the classical Yang-Mills equation, i.e. to instantons and merons, while others appear to play unprecedented roles. A remarkable new class of IR degrees of freedom consists of Faddeev-Niemi-type link and knot solutions, potentially related to glueballs.

References

1. T. Schaefer and E.V. Shuryak, *Rev. Mod. Phys.* **70** (1998) 323; D.I. Diakonov, *Prog. Part. Nucl. Phys.* **51** (2003) 173. For an introduction see H. Forkel, *A Primer on Instantons in QCD*, hep-ph/0009136.
2. M.N. Chernodub, M.I. Polikarpov, arXiv:hep-th/9710205; G. Bali, arXiv:hep-ph/9809351; R. Haymaker, *Phys. Rep.* **315** (1999) 153.
3. G. 't Hooft, in *High Energy Physics*, edited by A. Zichichi, Editrice Compositori, Bologna, (1976); S. Mandelstam, *Phys. Rep.* **23C** (1976) 245; G. 't Hooft, *Nucl. Phys.* **B190** (1981), 455; A. Kronfeld, G. Schierholz and U. J. Wiese, *Nucl. Phys.* **B293** (1987) 461.
4. G. 't Hooft, *Nucl. Phys.* **B138** (1978) 1; J. Greensite, *Prog. Part. Nucl. Phys.* **51** (2003) 1.
5. Yu. Makeenko and A.A. Migdal, *Nucl. Phys.* **B188** (1981) 269; A.A. Migdal, *Phys. Rep.* **102** (1983) 199; A. Dubin and Yu. Makeenko, in *At the frontier of particle physics*, vol. 4, p. 2479 Ed. M. Shifman, World Scientific, Singapore (2002).
6. N. Christ and T.D. Lee, *Phys. Rev.* **D22** (1980) 939.
7. H. Forkel, *Phys. Rev.* **D73** (2006) 105002.
8. R. Jackiw, in *“Field theory and particle physics”*, Eds. O.J.P. Eboli, M. Gomes, A. Santoro, World Scientific, Singapore 1990, p. 731.
9. I.I. Kogan and A. Kovner, *Phys. Rev.* **D52** (1995) 3719.
10. E.V. Shuryak and J.J.M. Verbaarschoot, *Phys. Rev.* **D52** (1995) 295.
11. C. Michael and P.S. Spencer, *Phys. Rev.* **D52** (1995) 4691; T. DeGrand, A. Hasenfratz and T.G. Kovacs, *Nucl. Phys.* **B505** (1997) 417; Ph. de Forcrand, M. Garcia Perez and I.-O. Stamatescu, *Nucl. Phys.* **B499** (1997) 409; M.J. Teper, *Phys. Rev.* **D58** (1998) 014505.
12. L.D. Faddeev, *Quantization of Solitons*, preprint IAS-75-QS70; L.D. Faddeev and A.J. Niemi, *Nature* **387** (1997) 58; *Phys. Rev. Lett.* **82** (1999) 1624.